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Global surpluses of spin-base invariant fermions



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ABSTRACT

The spin-base invariant formalism of Dirac fermions in curved space maintains the essential symmetries of general covariance as well as similarity transformations of the Clifford algebra. We emphasize the advantages of the spin-base invariant formalism both from a conceptual as well as from a practical viewpoint. This suggests that local spin-base invariance should be added to the list of (effective) properties of (quantum) gravity theories. We find support for this viewpoint by the explicit construction of a global realization of the Clifford algebra on a 2-sphere which is impossible in the spin-base non-invariant vielbein formalism.

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1. Introduction

The mutual interrelation of matter and spacetime (“matter curves spacetime – spacetime determines the paths of matter”) is particularly apparent for fermions. For instance for Dirac fermions, information about both spin as well as spacetime meets in the Clifford algebra,

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}I, \quad (1)$$

where the Dirac matrices γ_μ as well as the metric $g_{\mu\nu}$ generally are spacetime dependent. While many tests of classical gravity rely on vacuum solutions to Einstein's equation, also many attempts at quantizing gravity primarily concentrate on the dynamics of spacetime without matter, cf. [1]. This is similar in spirit to “quenched” QCD which allows to understand already many features of the strong interactions at the quantum level even quantitatively. Only recently, some evidence has been collected that the existence of matter degrees of freedom can constrain the existence of certain quantum gravity theories [2–6]. This is again analogous to QCD where the presence of too many dynamical fermions can destroy the high-energy completeness of the theory.

The interrelation of gravity and fermions provided by (1) has also been interpreted in various partly conflicting directions: read from right to left, one is tempted to conclude that one first needs a spacetime metric $g_{\mu\nu}$ in order to give a meaning to spinorial degrees of freedom and corresponding physical observables such as currents $\sim \bar{\psi}\gamma_\mu\psi$. On the other hand, representation theory of the Lorentz group in flat space suggests that all nontrivial representa-

tions can be composed out of the fundamental spinorial representation, culminating into (1) for Dirac spinors. If so, then also the metric might be a composite degree of freedom, potentially arising as an expectation value of composite spinorial operators, see, e.g., [7–9].

As a starting point to disentangle this hen-or-egg problem – spinors or metric first? – we consider the Clifford algebra (1) as fundamental in this work. We emphasize that this is different from a conventional approach [10], where one starts from the analogous Clifford algebra in flat (tangential) space, $\{\gamma_{(t)a}, \gamma_{(t)b}\} = 2\eta_{ab}I$, with fixed $\gamma_{(t)a}$ and then uplifts the Clifford algebra to curved space with the aid of a vielbein $e_\mu^a(x)$, such that $\gamma_{(e)\mu} = e_\mu^a\gamma_{(t)a}$ satisfies (1). In addition to diffeomorphism invariance, the vielbein approach supports a local $SO(3,1)$ symmetry of Lorentz transformations in tangential space, i.e. with respect to the roman *bein* index. By contrast, the Clifford algebra (1) actually supports a bigger symmetry of local similarity (spin-base) transformations in addition to general covariance.

Developing a formalism that features this full spin-base invariance has first been initiated by Schrödinger [11] and amended with the required spin metric by Bargmann [12] in 1932. Surprisingly, it has been rarely used in the literature, see, e.g., [13–19], or even reinvented [20]. A full account of the formalism also including spin torsion has recently been given in [21]. Particular advantages are not only the inclusion and generalization of the vielbein formalism. In a quantized setting, it even justifies the widespread use of the vielbein as an auxiliary quantity and not as a fundamental entity. Common quantization schemes relying on the metric as fundamental degree of freedom remain applicable also with fermionic matter. Hence, a Jacobian from the variable transformation to the vielbein does not have to be accounted for [21].

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In this work, we present further advantages of the spin-base invariant formalism and discuss some general aspects in order to elucidate the interplay between diffeomorphisms and spin-base transformations. We point out various options of defining the spin-base group, differing by the possible field content of further interactions and also naturally permitting a Spin^c structure. Since the conventional vielbein formalism can always be recovered within the spin-base invariant formalism, it is tempting to think that the latter is merely a technical perhaps overabundant generalization of the former. We demonstrate that this is not the case by an explicit construction of a global spin-base on the 2-sphere – a structure which is not possible in the conventional formalism because of global obstructions from the Poincaré–Brouwer (hairy-ball) theorem. We believe that this example is paradigmatic for the surpluses of the spin-base invariant formalism.

2. General covariance and spin-base invariance

Local symmetries are expected to be fundamental, since symmetry-breaking perturbations typically contain relevant components which inhibit symmetry emergence. Hence, we consider the local symmetries of the Clifford algebra as fundamental. These are diffeomorphisms (formalized by tensor calculus of the Greek indices) and local similarity transformations of the Dirac matrices [22], the spin base transformations,

$$\gamma_\mu \rightarrow S \gamma_\mu S^{-1}, \quad \psi \rightarrow S \psi, \quad \bar{\psi} \rightarrow \bar{\psi} S^{-1}. \quad (2)$$

The γ_μ transformation leaves the Clifford algebra equation (1) invariant. The corresponding transformation of spinors ensures that typical fermion bilinears and higher-order interaction terms serving as building blocks for a relativistic field-theory are also invariant, provided a suitable connection exists. The latter should obey

$$\Gamma_\mu \rightarrow S \Gamma_\mu S^{-1} - (\partial_\mu S) S^{-1}, \quad (3)$$

such that $\nabla_\mu = \partial_\mu + \Gamma_\mu$ forms a covariant derivative with the standard covariance properties with respect to both diffeomorphisms as well as spin base transformations. The connection Γ_μ has explicitly been constructed in $d = 4$ dimensions [20,21] as well as in lower [23] and higher dimensions [24]. For vanishing spin torsion [21], the traceless part of Γ_μ can fully be expressed in terms of the Dirac matrices and their first derivatives (part of the terms can be summarized by Christoffel symbols).

For simplicity, let us confine ourselves to the cases $d = 4$ and $d = 2$ (for generalizations, see [24]). Here, the dimension of the irreducible representation of the Clifford algebra is $d_\gamma = 4$ and $d_\gamma = 2$, respectively. A natural choice for the group of spin base transformations maintaining all invariance properties mentioned above is then given by $\text{GL}(d_\gamma, \mathbb{C})$.

However, $\text{GL}(d_\gamma, \mathbb{C})$ contains continuous subgroups that act trivially on the Clifford algebra. Considering the invariance properties of the Clifford algebra as fundamental, trivial subgroups appear redundant. Locally, elements of $\text{GL}(d_\gamma, \mathbb{C})$ can be decomposed into an $\text{SL}(d_\gamma, \mathbb{C})$ element and two factors proportional to the identity: a phase $\in \text{U}(1)$ and a modulus $\in \mathbb{R}_+$. Confining ourselves to the nontrivial invariance properties, hence suggest to identify the set of transformation matrices S with the fundamental representation of $\text{SL}(d_\gamma, \mathbb{C})$. This special linear group still has redundancies as its discrete center \mathbb{Z}_{d_γ} does not transform the Dirac matrices nontrivially.

The choice of the local spin-base group becomes only relevant, once a dynamics is associated with the connection. For the choice of $\text{SL}(d_\gamma, \mathbb{C})$ and vanishing torsion, the corresponding field strength $\Phi_{\mu\nu}$ satisfies the identity [20,21]

$$\Phi_{\mu\nu} = [\nabla_\mu, \nabla_\nu] = \frac{1}{8} R_{\mu\nu\lambda\kappa} [\gamma^\lambda, \gamma^\kappa]. \quad (4)$$

It is somewhat surprising as well as reassuring that – out of the large number of degrees of freedom in Γ_μ – only those acquire a nontrivial dynamics which can be summarized in the Christoffel symbols and hence lead to the Riemann tensor on the right-hand side of Eq. (4). As a consequence, spin-base invariance is also a (hidden) local symmetry of any special relativistic fermionic theory in flat space with an automatically trivial dynamics for the connection, even if kinetic terms of the form $\sim \text{tr} \gamma^\mu \Phi_{\mu\nu} \gamma^\nu$ ($\sim R$ Einstein–Hilbert) or $\sim \text{tr} \Phi_{\mu\nu} \Phi^{\mu\nu}$ would be added.

This is different if spin-base transformations are associated with $\text{GL}(d_\gamma, \mathbb{C})$. Then two additional abelian field strengths corresponding to the $\text{U}(1)$ and the non-compact \mathbb{R}_+ factors appear on the right-hand side of Eq. (4) and thus introduce further physical degrees of freedom. These correspond to the imaginary and real part of the trace of the connection Γ_μ .¹ Hence, the identification of the spin base group is in principle an experimental question to be addressed by verifying the interactions of fermions. In this sense, one might speculate whether the hypercharge $\text{U}(1)$ factor of the standard model could be identified with the spin-base group provided proper charge assignments are chosen for the different fermions. The inclusion of the $\text{U}(1)$ factor is particularly natural on manifolds that do not permit a Spin structure (e.g., CP^2) [25], as it provides exactly for the necessary ingredient to define the more general Spin^c structure.

For the remainder of this work, it suffices to consider $\text{SL}(d_\gamma, \mathbb{C})$ as the group of spin-base transformations. Returning to the hen-or-egg problem, Eq. (4) seems interpretable as another manifestation of the intertwining of Dirac structure and curvature, or spin-base and general covariance. However, a clearer picture arises from an explicit coordinate transformation of the Clifford algebra,

$$\{\gamma'_\mu, \gamma'_\nu\} = 2g'_{\mu\nu} \mathbf{I} = 2 \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\lambda}{\partial x'^\nu} g_{\rho\lambda} \mathbf{I} = \left\{ \frac{\partial x^\rho}{\partial x'^\mu} \gamma_\rho, \frac{\partial x^\lambda}{\partial x'^\nu} \gamma_\lambda \right\}. \quad (5)$$

Read together with the spin-base invariance of the Clifford algebra [22,26], Eq. (5) implies that the most general coordinate transformation of a Dirac matrix is given by

$$\gamma_\mu \rightarrow \gamma'_\mu = \frac{\partial x^\rho}{\partial x'^\mu} S \gamma_\rho S^{-1}. \quad (6)$$

From the sheer size of the spin-base group (at least $\text{SL}(d_\gamma, \mathbb{C})$), it is obvious that this is a larger set of Dirac matrices satisfying the Clifford algebra than can be spanned by the vielbein construction. In the latter, only those realizations of the Clifford algebra $\gamma_{(e)\mu}$ are considered, that can be spanned by a fixed set of Dirac matrices, $\gamma_{(e)\mu} = e_\mu^a \gamma_{(f)a}$. A local Lorentz transformation with respect to the *bein* index can then be rewritten in terms of

$$\Lambda_a^b \gamma_{(f)b} = S_{\text{Lor}} \gamma_{(f)a} S_{\text{Lor}}^{-1}, \quad (7)$$

where $S_{\text{Lor}} \in \text{Spin}(d-1, 1) \subset \text{SL}(d_\gamma, \mathbb{C})$. Conventionally, the S_{Lor} factors are interpreted as Lorentz transformations of Dirac spinors, e.g., $\psi \rightarrow S_{\text{Lor}} \psi$. This way of interpreting the Lorentz subgroup of spin-base transformations is at the heart of understanding fields as representations of the Lorentz group. This viewpoint is held to argue that higher-spin fields (such as the metric) may eventually be composed out of a fundamental spinorial representation.

However, there is no such simple relation as Eq. (7) for general coordinate transformations. This is already obvious in flat space:

¹ The non-compact factor (real part of $\text{tr} \Gamma_\mu$) can be removed by fixing the determinant of the spin metric, see [20,21].

rescaling one coordinate axis, say, $x^3 \rightarrow x^3/\alpha$, implies a change of the metric,

$$g_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \rightarrow g'_{\mu\nu} = \text{diag}(-1, 1, 1, \alpha^2). \quad (8)$$

The corresponding change of the Dirac matrices cannot be written purely in terms of a spin-base transformation, since γ'_3 has to satisfy $(\gamma'_3)^2 = \alpha^2 \mathbb{I}$, whereas $(S\gamma_3 S^{-1})^2 = \mathbb{I}$ for all $S \in \text{SL}(d_\gamma, \mathbb{C})$.

It is therefore more natural to view the general coordinate transformation (6) of the Dirac matrices as consisting of two independent transformations, (i) the change of the spacetime basis (diffeomorphisms), $\gamma_\mu \rightarrow \frac{\partial x^\rho}{\partial x'^\mu} \gamma_\rho$, and (ii) the change of the spin base equation (2). In particular, there is no need to intertwine these transformations.

While these statements seem self-evident within the present discussion, they may appear uncommon if compared to the conventional reasoning in flat space. Coordinate transformations between two different Lorentz frames, $\Lambda_a^b = \frac{\partial x^b}{\partial x'^a}$, are typically combined with spin-base transformations using Eq. (7) in order to keep the Dirac matrices in the new frame form-identical to those in the old frame, $\gamma_{(\Gamma)_a} \equiv \gamma_{(\Gamma)_a}$. This is, however, not necessary, as also $\gamma'_a = \Lambda_a^b \gamma_{(\Gamma)_b}$ satisfies the Clifford algebra.

To summarize, spinors should be viewed as objects that transform as scalars under diffeomorphisms and as “vectors” under spin-base transformations. In flat space, Lorentz transformations and spin-base transformations may be combined in order to keep the Dirac matrices fixed. We emphasize that the latter is merely a convenient choice and by no means mandatory. In fact, the freedom not to link the two transformations can have significant advantages as shown in the next section.

In view of the hen-or-egg problem, this symmetry analysis does not single out a specific viewpoint. On the one hand, the representation theory of the Lorentz group suggesting “spinors first” should be embedded into the larger spin-base invariant framework; while this presumably does not change the result for the classification of fields, there is no analogue of Eq. (7) for general spin-base transformations. On the other hand, the fact that we need a metric to define the Clifford algebra, does not link spinors closer to the metric as other fields; diffeomorphisms leave spinors untouched and the transformed Dirac matrices satisfy the Clifford algebra automatically. Instead, our analysis rather suggests that not only local Lorentz invariance, but full local spin-base invariance should be a requirement for possible underlying quantum theories of gravity. If not at the fundamental level, local spin-base invariance should at least be emergent for the long-range effective description.

3. Global spin base

It is a legitimate question as to whether spin-base invariance introduces an overabundant symmetry structure without gaining any advantages or further insights. In fact, already the vielbein formalism with much less symmetry has been criticized for its redundancy. For instance, the Ogievetsky–Polubarinov spinors [27] not only remove the $\text{SO}(3,1)$ redundancy of the vielbein formulation (analogous to the Lorentz symmetric gauge for the vielbein [28]), but make spinors compatible with tensor calculus, see, e.g., [29].

Nevertheless, $\text{SL}(d_\gamma, \mathbb{C})$ spin-base invariance is not a symmetry that may or may not be constructed on top of existing symmetries. On the contrary, global spin-base invariance is present in any relativistic fermionic theory. Its local version does not need an additional new compensator field, but the connection Γ_μ is built from the Dirac matrices which are present anyway. We will now present an example which demonstrates the advantages of full spin-base invariance.

Rather generically, smooth orientable manifolds may not be parametrizable with a single coordinate system, but may require several overlapping coordinate patches. In the vielbein formalism, where $g_{\mu\nu} = e_\mu^a \eta_{ab} e_\nu^b$, it is natural to expect that patches with different coordinates and corresponding metrics $g_{\mu\nu}$ also require different vielbeins e_μ^a . This becomes most obvious for the simple example of a 2-sphere which requires at least two overlapping coordinate patches to be covered. The same is true for the vielbein: for each fixed bein index, e_μ^a is a spacetime vector which has to satisfy the Poincaré–Brouwer (hairy-ball) theorem. This implies that it has to vanish at least on one point of the 2-sphere (such that also $\det e = 0$). Hence, at least two sets of vielbeins and corresponding transition functions are required to cover the 2-sphere without singularities.

For the spin-base invariant formalism, the independence of diffeomorphisms and spin-base transformations suggests that a change of the coordinate patch and metric does not necessarily require a change of the spin-base patch. More constructively, two sets of spin bases on two neighboring coordinate patches may be smoothly connected by a suitable spin-base transformation. We now show that this is possible for the 2-sphere resulting in a global spin base.

To keep this discussion transparent, we use the pair of polar and azimuthal angles (θ, ϕ) to label all points on the sphere (not as coordinates). For the polar coordinates, we use the notation (ϑ, φ) , i.e., $(x^\mu)|_{(\theta, \phi)} = (\vartheta, \varphi)|_{(\theta, \phi)} = (\theta, \phi)$. These are legitimate coordinates except for the poles at $\theta \in \{0, \pi\}$. In polar coordinates the metric reads

$$(g_{\mu\nu})|_{(\theta, \phi)} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \vartheta \end{pmatrix} \Big|_{(\theta, \phi)} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}. \quad (9)$$

Obviously, the metric becomes degenerate at the poles, rendering the coordinates ill-defined there. In these coordinates, one suitable choice for the vielbein e_μ^a is

$$(e_\mu^a)|_{(\theta, \phi)} = \begin{pmatrix} 1 & 0 \\ 0 & \sin \vartheta \end{pmatrix} \Big|_{(\theta, \phi)} = \begin{pmatrix} 1 & 0 \\ 0 & \sin \theta \end{pmatrix}. \quad (10)$$

This choice is perfectly smooth everywhere, but is not appropriate at the poles. In order to cover the poles $\theta \in \{0, \pi\}$, we need to change coordinates. For simplicity, we choose Cartesian coordinates $(x'^\mu)|_{(\theta, \phi)} = (x, y)|_{(\theta, \phi)} = (\cos \phi \sin \theta, \sin \phi \sin \theta)$, these are well defined at the poles but ill defined at the equator. For the coordinate transformation, we need the Jacobian

$$\left(\frac{\partial x^\nu}{\partial x'^\mu} \right) \Big|_{(\theta, \phi)} = \begin{pmatrix} \frac{\cos \phi}{\cos \theta} & -\frac{\sin \phi}{\sin \theta} \\ \frac{\sin \phi}{\cos \theta} & \frac{\cos \phi}{\sin \theta} \end{pmatrix}. \quad (11)$$

We emphasize again that the pair (θ, ϕ) is used only for convenience to label a point on the sphere and not as a coordinate system. The metric for the primed (Cartesian) coordinates x'^μ reads

$$(g'_{\mu\nu})|_{(\theta, \phi)} = \frac{1}{1 - x^2 - y^2} \begin{pmatrix} 1 - y^2 & xy \\ xy & 1 - x^2 \end{pmatrix} \Big|_{(\theta, \phi)} \quad (12)$$

$$= \frac{1}{\cos^2 \theta} \begin{pmatrix} 1 - \sin^2 \phi \sin^2 \theta & \sin \phi \cos \phi \sin^2 \theta \\ \sin \phi \cos \phi \sin^2 \theta & 1 - \cos^2 \phi \sin^2 \theta \end{pmatrix}. \quad (13)$$

The transformed vielbein $e'_\mu{}^a$ yields

$$(e'_\mu{}^a)|_{(\theta, \phi)} = \left(\frac{\partial x^\nu}{\partial x'^\mu} e_\nu^a \right) \Big|_{(\theta, \phi)} = \frac{1}{\cos \theta} \mathcal{R}_\phi \begin{pmatrix} 1 & 0 \\ 0 & \cos \theta \end{pmatrix}, \quad (14)$$

$$\mathcal{R}_\phi = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}. \quad (15)$$

First, we observe a coordinate singularity at the equator as expected. Moreover, we obtain a ϕ dependence which seems to render the vielbein ill defined at the poles. Nevertheless, this can be cured by performing a corresponding (counter-)rotation in tangential space with respect to the bein index. The hairy ball theorem manifests itself here by the fact that one pole needs a rotation, while the other needs a combination of the same rotation and a reflection. These are elements of the two different connected components of the rotation group $O(2)$, respectively, the proper and improper rotations. Since we cannot perform a continuous transformation from proper to improper rotations, we cannot cure the residual ϕ dependence at both poles at the same time in a continuous way (independently of the expected coordinate singularity at the equator). Incidentally, an inverse rotation would also cure the problematic ϕ dependence at the south pole; but because of the required 2π periodicity in ϕ , the direction of the rotation cannot be changed continuously from north to south pole. The same conclusion remains true for those sets of Dirac matrices which are constructed via the vielbein $\gamma'_{(e)\mu}|_{(\theta,\phi)} = e'^\mu_a|_{(\theta,\phi)} \gamma_{(f)a}$.

By contrast, the spin-base invariant formalism allows to continuously connect all representations of the two dimensional Clifford algebra, i.e., proper and improper rotations of $O(2)$ should be continuously connectable on the level of $SL(2, \mathbb{C})$ spin-base transformations of the Dirac matrices.

For this, we first define conventional *constant* flat Dirac matrices using the Pauli matrices

$$(\tilde{\gamma}_{(f)a}) = \begin{pmatrix} \sigma_1 \\ -\sigma_2 \end{pmatrix}, \quad (16)$$

which fulfill the two dimensional flat Euclidean Clifford algebra $\{\tilde{\gamma}_{(f)a}, \tilde{\gamma}_{(f)b}\} = 2\delta_{ab}I$. Next, we construct auxiliary spacetime *dependent* flat Dirac matrices,

$$\gamma_{(f)a}|_{(\theta,\phi)} = \mathcal{S}(\theta, \phi) \tilde{\gamma}_{(f)a} \mathcal{S}^{-1}(\theta, \phi), \quad (17)$$

$$\mathcal{S}(\theta, \phi) = e^{-i\frac{\phi}{2}\sigma_3} e^{-i\frac{\theta-\pi}{2}\sigma_1}, \quad (18)$$

which also satisfy the Euclidean Clifford algebra as Eq. (17) is a spin-base transformation. We emphasize that Eq. (18) goes beyond the subgroup of $Spin(2)$ transformations because of the second exponential factor. The new flat Dirac matrices read explicitly

$$\gamma_{(f)1}|_{(\theta,\phi)} = \cos\phi\sigma_1 + \sin\phi\sigma_2, \quad (19)$$

$$\gamma_{(f)2}|_{(\theta,\phi)} = \cos\theta(-\sin\phi\sigma_1 + \cos\phi\sigma_2) + \sin\theta\sigma_3. \quad (20)$$

Here it becomes manifest, that these Dirac matrices smoothly vary from a proper rotation at $\theta = 0$ to an improper rotation at $\theta = \pi$, while maintaining 2π -periodicity in ϕ . Based on this special set of flat-space Dirac matrices, the Dirac matrices on the 2-sphere in Cartesian coordinates $\gamma'_\mu = e'^\mu_a \gamma_{(f)a}$ read

$$(\gamma'_\mu|_{(\theta,\phi)}) = \frac{1}{\cos\theta} \mathcal{R}_\phi \begin{pmatrix} 1 & 0 \\ 0 & \cos^2\theta \end{pmatrix} \mathcal{R}_\phi^{-1} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} + \sin\theta \mathcal{R}_\phi \begin{pmatrix} 0 \\ \sigma_3 \end{pmatrix}. \quad (21)$$

These Dirac matrices are obviously well behaved at the poles $\theta \in \{0, \pi\}$, since there are no singularities and no ϕ dependence is left. Of course, the singularity at the equator remains, where the Cartesian coordinates are ill defined. This singularity is not present in polar coordinates, where we obtain the Dirac matrices $\gamma_\mu = \frac{\partial x'^\rho}{\partial x^\mu} \gamma'_\rho$

$$(\gamma_\mu|_{(\theta,\phi)}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2}\sin 2\theta \end{pmatrix} \mathcal{R}_\phi^{-1} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} + \sin^2\theta \begin{pmatrix} 0 \\ \sigma_3 \end{pmatrix}. \quad (22)$$

Note that γ_μ and γ'_μ are connected solely by a diffeomorphism – no change of the spin base is involved. Whereas the vielbein construction given above actually proceeded via ill-defined intermediate objects,² the resulting spin base chosen for the curved Dirac matrices given by Eq. (21) in Cartesian coordinates (i.e. except for the equator) and by Eq. (22) in polar coordinates (i.e. except for the poles) holds globally all over the 2-sphere. No additional patch for spin-base coordinates is required to cover the whole 2-sphere. In particular the limit towards the poles in Eq. (21) is unique and smooth in contrast to the vielbein case.

It is interesting to see how the spin-base invariant formalism evades the hairy-ball theorem: the important point is that γ_μ does not represent a globally non-vanishing vector field (which would be forbidden), but is a vector of Dirac matrix fields, $(\gamma_\mu)^I_J$. For every fixed pair $(I, J) \in \{1, \dots, d_\gamma\}^2$, we have a complex vector field. It is easy to check that each of the real sub-component vector fields has at least one zero on the sphere, being therefore compatible with the hairy-ball theorem. The zeros of these vector fields are however distributed such that the Dirac matrices γ_μ satisfy the Clifford algebra all over the 2-sphere.

We expect that the construction above generalizes to all $2n$ -spheres, since the corresponding spin-base group $SL(d_\gamma, \mathbb{C})$ with $d_\gamma = 2^n$ is connected and all representations of the Dirac matrices are connected to each other via a spin-base transformation. The problem of the disconnected components of the orthogonal group should then be resolvable in the same way as shown above. Incidentally, the hairy-ball theorem applies to the $2n$ -spheres, implying that vielbeins cannot be defined globally on these spheres.

As an application of this global spin base, let us study the eigenfunctions of the Dirac operator on the 2-sphere. Using the vielbein e_μ^a of Eq. (10) and the flat Dirac matrices $\tilde{\gamma}_{(f)a}$ of Eq. (16) the eigenfunctions have been calculated in [30] within the vielbein formalism. The vielbein spin connection $\Gamma_{(e)\mu} = \frac{1}{8}\omega_{\mu}^{ab}[\tilde{\gamma}_{(f)a}, \tilde{\gamma}_{(f)b}]$ in spherical coordinates is then given by

$$(\Gamma_{(e)\mu}|_{(\theta,\phi)}) = \begin{pmatrix} 0 \\ \frac{i}{2}\cos\theta\sigma_3 \end{pmatrix}. \quad (23)$$

The eigenfunctions of the Dirac operator $\not{D} = \gamma_{(e)\mu}(\partial_\mu + \Gamma_{(e)\mu})$ satisfy $\not{D}\psi_{\pm,n,l}^{(s)} = \pm i(n+1)\psi_{\pm,n,l}^{(s)}$, $s \in \{-1, 1\}$ and read [30]

$$\psi_{\pm,n,l}^{(-)}(\theta, \phi) = \frac{c_2(n, l)}{\sqrt{4\pi}} e^{-i(l+\frac{1}{2})\phi} \begin{pmatrix} \Phi_{n,l}(\theta) \\ \pm i\Psi_{n,l}(\theta) \end{pmatrix}, \quad (24)$$

$$\psi_{\pm,n,l}^{(+)}(\theta, \phi) = \frac{c_2(n, l)}{\sqrt{4\pi}} e^{i(l+\frac{1}{2})\phi} \begin{pmatrix} i\Psi_{n,l}(\theta) \\ \pm \Phi_{n,l}(\theta) \end{pmatrix}, \quad (25)$$

where $n \in \mathbb{N}_0$, $l \in \{0, \dots, n\}$, and

$$\Phi_{n,l}(\theta) = \cos^{l+1}\frac{\theta}{2} \sin^l\frac{\theta}{2} P_{n-l}^{(l,l+1)}(\cos\theta), \quad (26)$$

$$\Psi_{n,l}(\theta) = (-1)^{n-l}\Phi_{n,l}(\pi - \theta), \quad (27)$$

$$c_2(n, l) = \frac{\sqrt{(n-l)!(n+l+1)!}}{n!}, \quad (28)$$

with the Jacobi polynomials $P_n^{(\alpha,\beta)}$. The following properties of the eigenfunction deserve particular attention:

$$\psi_{\pm,n,l}^{(s)}(\theta, \phi + 2\pi) = -\psi_{\pm,n,l}^{(s)}(\theta, \phi), \quad (29)$$

$$\psi_{\pm,n,l=0}^{(s)}(\theta = 0, \phi) = \sqrt{\frac{n+1}{16\pi}} e^{i\frac{\phi}{2}} \begin{pmatrix} 1-s \\ \pm(1+s) \end{pmatrix}, \quad (30)$$

² With hindsight, the ill-definiteness of $\gamma_{(e)\mu}|_{(\theta,\phi)}$ at the poles is cured by the properties of the flat gamma matrix $\gamma_{(f)a}|_{(\theta,\phi)}$ which are analogously ill defined at the poles.

$$\psi_{\pm,n,l=0}^{(s)}(\theta = \pi, \phi) = i(-1)^n \sqrt{\frac{n+1}{16\pi}} e^{i\frac{\phi}{2}} \begin{pmatrix} 1+s \\ \pm(1-s) \end{pmatrix}. \quad (31)$$

Eq. (29) shows that the eigenspinors pick up a minus sign upon a 2π rotation. Eqs. (30), (31) reveal that the eigenspinors are not well defined at the poles, as an ambiguous ϕ dependence remains. This is similar to the residual ϕ dependence of the vielbein at the poles.

Let us now study these properties with the global spin base constructed above. The Dirac matrices γ_μ of Eq. (22) and $\gamma_{(e)\mu}$ are connected via the spin-base transformation (18), $\gamma_\mu = \mathcal{S}\gamma_{(e)\mu}\mathcal{S}^{-1}$. The corresponding spin connection Γ_μ can be calculated from

$$\Gamma_\mu = \mathcal{S}\Gamma_{(e)\mu}\mathcal{S}^{-1} - (\partial_\mu\mathcal{S})\mathcal{S}^{-1}, \quad (32)$$

leading to $\Gamma_\mu = \frac{i}{2}\gamma_\mu$. The eigenfunctions $\hat{\psi}_{\pm,n,l}^{(s)}$ of the Dirac operator $\not{\nabla} = \gamma^\mu(\partial_\mu + \Gamma_\mu)$ in the global spin base are then given by

$$\hat{\psi}_{\pm,n,l}^{(s)}(\theta, \phi) = \mathcal{S}(\theta, \phi)\psi_{\pm,n,l}^{(s)}(\theta, \phi). \quad (33)$$

It is now straightforward to check that these eigenfunctions are globally well behaved, in particular

$$\hat{\psi}_{\pm,n,l}^{(s)}(\theta, \phi + 2\pi) = \hat{\psi}_{\pm,n,l}^{(s)}(\theta, \phi), \quad (34)$$

$$\hat{\psi}_{\pm,n,l=0}^{(s)}(\theta = 0, \phi) = i\sqrt{\frac{n+1}{16\pi}} \begin{pmatrix} \pm(1+s) \\ 1-s \end{pmatrix}, \quad (35)$$

$$\hat{\psi}_{\pm,n,l=0}^{(s)}(\theta = \pi, \phi) = i(-1)^n \sqrt{\frac{n+1}{16\pi}} \begin{pmatrix} 1+s \\ \pm(1-s) \end{pmatrix}. \quad (36)$$

Not only has the ambiguous ϕ dependence disappeared at the poles, but also the spinors have become 2π -periodic. Since the eigenfunctions form a complete set of spinor functions on the 2-sphere, we have found a spin base that permits to span functions on the sphere in terms of globally defined smooth base spinors. This can serve as a convenient starting point for the construction of functional integrals for quantized fermion fields.

4. Conclusion

We have emphasized the importance of spin-base invariance for the description of fermionic degrees of freedom in relativistic theories. Local $SL(d_\gamma, \mathbb{C})$ spin-base invariance is a (hidden) symmetry of relativistic theories without adding any new propagating gauge degrees of freedom to the theory in flat space. In curved space, the associated dynamical degrees of freedom exactly correspond to those of general relativity. Whereas general covariance and spin-base invariance seem hardwired to each other via the Clifford algebra, we have stressed in this work that the associated symmetry transformations can be used fully independent of each other.

We have demonstrated this explicitly by constructing a global spin base on a 2-sphere which does not permit an equally globally well-defined choice of space coordinates. In other words, the coordinate patches required to cover a manifold do not have to be in one-to-one correspondence with the spin-base patches that cover the spinor space at all points of the manifold.

We consider this mutual independence of general covariance and spin-base invariance as an indication for the fact that the metric should not be viewed as more fundamental than the spin structure or vice versa. Both symmetries should therefore be a direct or emergent property of a more fundamental theory for matter and gravity.

In [21], we have already shown that quantum gravity theories that quantize the metric (e.g., in terms of a functional integral over $g_{\mu\nu}$) preserve spin-base invariance. This includes, for instance, the asymptotic safety scenario [31] for quantum Einstein gravity [32]. The mutual independence of general covariance and spin-base invariance is also a reason for the fact that a quantization of the spin-base degrees of freedom is not necessary [21], though certainly possible and legitimate [5,33].

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